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SYMMETRIC BI-f-MULTIPLIERS OF INCLINE ALGEBRAS

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Abstract. In this paper, we introduce the concept of a symmetric bi-f-multiplier in incline algebras and give some properties of incline algebras. Also, we characterize $Ker(D)$ and $Fix_a(D)$ by symmetric bi-f-multipliers in incline algebras.

1. Introduction

Z. Q. Cao, K. H. Kim and F. W. Roush [2] introduced the notion of incline algebras in their book. Some authors studied incline algebras and application. N. O. Alshehri [1] introduced the notion of derivation in incline algebras. In this paper, we introduce the concept of a symmetric bi-f-derivation in incline algebra and give some properties of incline algebras. Also, we characterize $Ker_D(K)$ and $Fix_D(K)$ by symmetric bi-f-derivations in incline algebras.

2. Incline algebras

An *incline algebra* is a set K with two binary operations denoted by " $+$ " and " $*$ " satisfying the following axioms:

(K1) $x + y = y + x$,
(K2) $x + (y + z) = (x + y) + z$,
(K3) $x * (y * z) = (x * y) * z$,
(K4) $x * (y + z) = (x * y) + (x * z)$,
(K5) $(y + z) * x = (y * x) + (z * x)$,
(K6) $x + x = x$,

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(K7) $x + (x * y) = x$,
(K8) $y + (x * y) = y$

for all $x, y, z \in K$.

For convenience, we pronounce " $+$ " (resp. " $*$ ") as addition (resp. multiplication). Every distributive lattice is an incline algebra. An incline algebra is a distributive lattice if and only if $x * x = x$ for all $x \in K$. Note that $x \leq y \Leftrightarrow x + y = y$ for all $x, y \in K$. It is easy to see that " $\leq$ " is a partial order on K and that for any $x, y \in K$, the element $x + y$ is the least upper bound of $\{x, y\}$. We say that $\leq$ is induced by operation $+$.

In an incline algebra K, the following properties hold.

(K9) $x * y \leq x$ and $y * x \leq x$ for all $x, y \in K$,
(K10) $y \leq z$ implies $x * y \leq x * z$ and $y * x \leq z * x$, for all $x, y, z \in K$,
(K11) If $x \leq y$ and $a \leq b$, then $x + a \leq y + b$, and $x * a \leq y * b$ for all $x, y, a, b \in K$.

Furthermore, an incline algebra K is said to be *commutative* if $x * y = y * x$ for all $x, y \in K$. A map f is *isotone* if $x \leq y$ implies $f(x) \leq f(y)$ for all $x, y \in K$.

A *subincline* of an incline algebra K is a non-empty subset M of K which is closed under the addition and multiplication. A subincline M is said to be an *ideal* if $x \in M$ and $y \leq x$ then $y \in M$. An element "0" in an incline algebra K is a *zero element* if $x + 0 = x = 0 + x$ and $x * 0 = 0 = 0 * x$ for any $x \in K$. An non-zero element "1" is called a *multiplicative identity* if $x * 1 = 1 * x = x$ for any $x \in K$. A non-zero element $a \in K$ is said to be a *left* (resp. *right*) *zero divisor* if there exists a non-zero $b \in K$ such hat $a * b = 0$ (resp. $b * a = 0$) A zero divisor is an element of K which is both a left zero divisor and a right zero divisor. An incline algebra K with multiplicative identity 1 and zero element 0 is called an *integral incline* if it has no zero divisors. By a *homomorphism* of inclines, we mean a mapping f from an incline algebra K into an incline algebra L such that $f(x + y) = f(x) + f(y)$ and $f(x * y) = f(x) * f(y)$ for all $x, y \in K$. A map $f : K \to K$ is *regular* if $f(0) = 0$. A subincline I of an incline algebra K is said to be *k-ideal* if $x + y \in I$ and $y \in I$, then $x \in I$. Let K be an incline algebra. An element $a \in K$ is called a *additively cancellative* if for all $a, b \in K$, $a + b = a + c \Rightarrow b = c$. If every element of K is additively cancellative, it is called *additively cancellative*.

Definition 2.1. Let K be an incline algebra. A mapping $D(.,.) : K \times K \to K$ is called *symmetric* if $D(x, y) = D(y, x)$ holds for all $x, y \in K$.

Definition 2.2. Let K be an incline algebra and $x \in K$. A mapping $d(x) = D(x, x)$ is called *trace* of $D(.,.)$, where $D(.,.) : K \times K \to K$ is a symmetric mapping.

Definition 2.3. Let K be an incline algebra and let $D : K \times K \to K$ be a symmetric mapping. We call D a symmetric bi-derivation on K if it satisfies the following condition

$$D(x * y, z) = (D(x, z) * y) + (x * D(y, z))$$

for all $x, y, z \in K$.

3. $*$-Symmetric bi-f-multipliers of incline algebras

In what follows, let K denote an incline algebra with a zero-element unless otherwise specified.

Definition 3.1. Let K be an incline algebra and let $D : K \times K \to K$ be a symmetric mapping. We call D a $*$-symmetric bi-f-multiplier on K if there exists a function $f : K \to K$ such that

$$D(x * y, z) = D(x, z) * f(y)$$

for all $x, y, z \in K$.

Obviously, a $*$-symmetric bi-f-multiplier D on K satisfies the relation

$$D(x, y * z) = D(x, y) * f(z)$$

for all $x, y, z \in K$.

Example 3.2. Let K be a commutative incline algebra. Define a mapping on K by $D(x, y) = f(x) * f(y)$ where $f : K \to K$ satisfies $f(x * y) = f(x) * f(y)$ for all $x, y \in K$. Then we can see that D is a $*$-symmetric bi-f-multiplier on K.

Example 3.3. Let K be a commutative incline algebra and $a \in K$. Define a mapping on K by $D(x, y) = (f(x) * f(y)) * a$ where $f : K \to K$ satisfies $f(x * y) = f(x) * f(y)$ for all $x, y \in K$. Then we can see that D is a $*$-symmetric bi-f-multiplier on K.

Example 3.4. Let $K = \{0, a, b, 1\}$ be a set in which "+" and "$*$" is defined by

$+$	0	a	b	1
0	0	a	b	1
a	a	a	b	1
b	b	b	b	1
1	1	1	1	1

$*$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

Then it is easy to check that $(K, +, *)$ is an incline algebra. Define a map $D : K \times K \to K$ by

$$D(x, y) = \begin{cases} b & \text{if } (x, y) \in \{(b, b), (b, 1), (1, b), (1, 1)\} \\ 0 & \text{otherwise} \end{cases}$$

and $f : K \to K$ by

$$f(x) = \begin{cases} b & \text{if } x \in \{b, 1\} \\ 0 & \text{otherwise} \end{cases}$$

Then it is easily checked that D is a $*$-symmetric bi-f-multiplier of an incline algebra K.

PROPOSITION 3.5. *Let K be an incline algebra and let D be a $*$-symmetric bi-f-multiplier on K. Then the following identities hold.*
(i) $D(x * y, z) \le f(y)$, for all $x, y, z \in K$,
(ii) $D(x, y) = D(x, y) * f(1)$, for all $x, y \in K$,
(iii) $D(x * y, z) \le D(x, z) + f(y)$, for all $x, y \in K$.

Proof. (i) Let $x, y, z \in K$. By using (K9), we have $D(x * y, z) = D(x, z) * f(y) \le f(y)$.

(ii) Let $x, y \in K$. Then we have $D(x, y) = D(x*1, y) = D(x, y)*f(1)$.

(iii) Let $x, y, z \in K$. Then we have $D(x * y, z) = D(x, z) * f(y) \le D(x, z)$. Also, we get $D(x, z) * f(y) \le f(y)$. Therefore, we have $D(x * y, z) \le D(x, z) + f(y)$. □

PROPOSITION 3.6. *Every $*$-symmetric bi-f-multiplier on K with $f(0) = 0$ is regular.*

Proof. Let D be a $*$-symmetric bi-f-multiplier on K with a zero element. Then we have

$$\begin{aligned} D(0, 0) &= D(x * 0, 0) = D(x, 0) * f(0) \\ &= D(x, 0) * 0 = 0 \end{aligned}$$

for all $x \in K$. □

PROPOSITION 3.7. *Let D be a $*$-symmetric bi-f-multiplier on K. If K is a distributive lattice, we have $D(x, y) \le f(x)$ and $D(x, y) \le f(y)$ for all $x, y \in K$.*

Proof. Let D be a $*$-symmetric bi-f-multiplier on K and let K be a distributive lattice. Then $D(x,y) = D(x*x,y) = D(x,y)*f(x)$, and so by using (K9), we get $D(x,y) \leq f(x)$. Similarly, we have $D(x,y) \leq f(y)$. □

PROPOSITION 3.8. *Let D be a $*$-symmetric bi-f-multiplier on K and let K be a distributive lattice. Then we have $d(x) \leq f(x)$ for all $x \in K$.*

Proof. Let D be a $*$-symmetric bi-f-multiplier on K and let K be a distributive lattice. Then we have

$$\begin{aligned} d(x) &= D(x,x) = D(x*x,x) = D(x,x)*f(x) \\ &= D(x,x)*f(x) \leq f(x) \end{aligned}$$

for all $x \in K$. □

THEOREM 3.9. *Let K be an integral incline with a multiplicative identity and let D be a $*$-symmetric bi-f-multiplier on K where f is a function satisfying $f(1) = 1$ and $a \in K$. Then for all $x, y \in K$, we have $D(x,y)*a = 0$ implies $a = 0$ or $D = 0$.*

Proof. Let $D(x,y)*a = 0$ for all $x, y \in K$. Since K is an integral incline, that is, it has no zero-divisors, we have $a = 0$ or $D(x,y) = 0$ for all $x, y \in K$. Hence we get $a = 0$ or $D = 0$. □

DEFINITION 3.10. Let K be an incline algebra. If $D : K \times K \to K$ be a symmetric mapping. We call D a *additive mapping* if it satisfies

$$D(x+y,z) = D(x,z) + D(y,z)$$

for all $x, y, z \in K$.

PROPOSITION 3.11. *Let d be a trace of additive $*$-symmetric bi-f-multiplier D on K. Then the following identities hold for all $x, y \in K$,*
(i) $d(x+y) = d(x) + d(y) + D(x,y)$ and $d(x) + d(y) \leq d(x+y)$,
(ii) $D(x*y,x) \leq d(x)$.

Proof. (i) Let $x, y \in K$. Then we have

$$\begin{aligned} d(x+y) &= D(x+y,x+y) = D(x,x+y) + D(y,x+y) \\ &= D(x,x) + D(x,y) + D(y,x) + D(y,y) \\ &= D(x,x) + D(y,y) + D(x,y). \end{aligned}$$

Hence we get $d(x+y) = d(x)+d(y)+D(x,y)$ and $d(x)+d(y) \leq d(x+y)$.
(ii) Let $x, y \in K$. It follows from (K7) that $d(x) = D(x,x) = D(x+(x*y),x) = D(x,x) + D(x*y,x)$, which implies $D(x*y,x) \leq d(x)$. □

Proposition 3.12. *Let D be a trace of $*$-symmetric bi-f-multiplier on K. Then $D(x*y,y) \leq D(x,y)$ for all $x,y \in K$.*

Proof. Let $x,y \in K$. Then we have

$$D(x,y) = D(x + x*y, y) = D(x,y) + D(x*y,y),$$

which implies $D(x*y,y) \leq D(x,y)$. □

Definition 3.13. Let D be a $*$-symmetric bi-f-multiplier on K. If $x \leq w$ implies $D(x,y) \leq D(w,y)$, D is called an *isotone $*$-symmetric bi-f-multiplier* for all $x,y,w \in K$.

Theorem 3.14. *Let D be a additive $*$-symmetric bi-f-multiplier on K. Then D is an isotone $*$-symmetric bi-f-multiplier on K.*

Proof. Let x and w be such that $x \leq w$. Then $x + w = w$, and so

$$D(w,y) = D(w + x, y) = D(w,y) + D(x,y)$$

for all $x,y,w \in K$. This implies that $D(x,y) \leq D(w,y)$. This completes the proof. □

Let D be a $*$-symmetric bi-f-multiplier on K and a be fixed element in K. Define a set $Fix_a(D) = \{x \in K | D(a,x) = f(x)\}$ for all $x \in K$.

Proposition 3.15. *Let D be a additive $*$-symmetric bi-f-multiplier and let f be an endomorphism on K. Then $Fix_a(D)$ is a subincline of K.*

Proof. Let $x,y \in Fix_a(D)$. Then we have $D(x,a) = f(x)$ and $D(y,a) = f(y)$, and so

$$\begin{aligned} D(x*y,a) &= D(x,a) * f(y) \\ &= f(x) * f(y) = f(x*y). \end{aligned}$$

Hence we get $x*y \in Fix_a(D)(K)$. Also, we get $D(x+y,a) = D(x,a) + D(y,a) = f(x)+f(y) = f(x+y)$, and so $x+y \in Fix_a(D)$. This completes the proof. □

Proposition 3.16. *Let D be a $*$-symmetric bi-f-multiplier on K with $f(x*y) = f(x)*f(y)$ for all $x,y \in K$. If $x \in Fix_a(D)$ and let f be an endomorphism on K, then $x*y \in Fix_a(D)$.*

Proposition 3.17. *Let K be additively cancellative and let D be a additive $*$-symmetric bi-f-multiplier on K and let f be an endomorphism on K. Then $Fix_a(D)$ is a k-ideal of K.*

Proof. Let $x+y \in Fix_a(D)$ and $y \in Fix_D(K)$. Then we have $f(x)+f(y) = f(x+y) = D(x+y,a) = D(x,a)+D(y,a) = D(x,a)+f(y)$. Since K is additively cancellative,we have $f(x) = D(x,a)$, which implies $x \in Fix_a(D)$. This completes the proof. □

DEFINITION 3.18. Let K be an incline algebra and let $D : K\times K \to K$ be a symmetric mapping. Define a set $Ker(D)$ by

$$Ker(D) = \{x \in K \mid D(0,x) = 0\}.$$

PROPOSITION 3.19. *Let D be a additive $*$-symmetric bi-f-multiplier on K. If $x \leq y$ and $y \in Ker(D)$, then we have $x \in Ker(D)$.*

Proof. Let $x \leq y$ and $y \in Ker(D)$. Then we get $x + y = y$ and $D(0,y) = 0$. Hence we get

$$\begin{aligned} 0 = D(0,y) &= D(0,x+y) \\ &= D(0,x)+D(0,y) \\ &= D(0,x)+0 = D(0,x), \end{aligned}$$

which implies $x \in Ker(D)$. This completes the proof. □

PROPOSITION 3.20. *Let D be a additive $*$-symmetric bi-f-multiplier on K. Then $Ker(D)$ is a subincline of K.*

Proof. Let $x,y \in Ker(D)$. Then $D(x,0) = 0$, and so

$$\begin{aligned} D(0,x*y) = D(x*y,0) &= D(x,0)*f(y) \\ &= 0*f(y) = 0, \end{aligned}$$

which implies $x*y \in Ker(D)$. Now $D(x+y,0) = D(x,0)+D(y,0) = 0+0 = 0$. Hence $x+y \in Ker(D)$. This completes the proof. □

THEOREM 3.21. *Let D be a additive $*$-symmetric bi-f-multiplier on K. Then $Ker(D)$ is an ideal of K.*

Proof. By Proposition 3.10 and 11, It is obvious that $Ker(D)$ is an ideal of K. □

4. $+$-Symmetric bi-f-multipliers of incline algebras

DEFINITION 4.1. Let K be an incline algebra and let $D : K\times K \to K$ be a symmetric mapping. We call D a *$+$-symmetric bi-f-multiplier* on K if there exists a function $f : K \to K$ such that

$$D(x,y+z) = D(x,y)+f(z)$$

for all $x,y,z \in K$.

EXAMPLE 4.2. Let K be an incline algebra. Define a mapping on K by $D(x,y) = x + f(y)$ where $f : K \to K$ satisfies $f(x+y) = f(x) + f(y)$ for all $x, y \in K$. Then we can see that D is a $+$-symmetric bi-f-multiplier on K.

PROPOSITION 4.3. *Let D be a $+$-symmetric bi-f-multiplier on K. Then the following identities hold.*
(i) $f(y) \leq D(x,y)$, for all $x, y, z \in K$,
(ii) $D(x,y) + f(y) \leq D(x,y)$, for all $x, y \in K$.

Proof. (i) Let D be a $+$-symmetric bi-f-multiplier on K. Then we have

$$D(x,y) = D(x, y+y) = D(x,0) + f(y),$$

which implies $f(y) \leq D(x,y)$.

(ii) Let D be a $+$-symmetric bi-f-multiplier on K. Then we have

$$D(x,y) = D(x, 0+y) = D(x,0) + f(y),$$

which implies $D(x,0) + f(y) \leq D(x,y)$. □

PROPOSITION 4.4. *Let D be a $+$-symmetric bi-f-multiplier on K with $f(x+y) = f(x) + f(y)$ for all $x, y \in K$. If $x \in Fix_a(D)$, then $x + y \in Fix_a(D)$ for all $y \in K$.*

Proof. Let D be a $+$-symmetric bi-f-multiplier on K and $x \in Fix_a(D)$. Then we have $D(a,x) = f(x)$. Hence

$$\begin{aligned} D(a, x+y) &= D(a,x) + f(y) = f(x) + f(y) \\ &= f(x+y), \end{aligned}$$

which implies $x + y \in Fix_D(K)$. □

PROPOSITION 4.5. *Let D be a $+$-symmetric bi-f-multiplier on an incline algebra K that is additively cancellative. If $f(x+y) = f(x)+f(y)$ for all $x, y \in K$ and $x + y \in Fix_a(D)$ and $y \in Fix_a(D)$, then $x \in Fix_a(D)$.*

Proof. Let D be a $+$-symmetric bi-f-multiplier and $x+y \in Fix_a(D)$. Then

$$\begin{aligned} f(x) + f(y) = f(x+y) &= D(a, x+y) \\ &= D(a,x) + f(y) \end{aligned}$$

Therefore we get $D(a,x) + f(y) = f(x) + f(y)$. Since K is additively cancellative, we have $D(a,x) = f(x)$, which implies $x \in Fix_a(D)$. □

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